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Multiobjective optimization problems with modified objective functions and cone constraints and applications

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Abstract In this paper, we consider a differentiable multiobjective optimization problem with generalized cone constraints (for short, MOP). We investigate the relationship between weakly efficient solutions for (MOP) and for the multiobjective optimization problem with the modified objective function and cone constraints [for short, $(\text{MOP})_\eta(x)$] and saddle points for the Lagrange function of $(\text{MOP})_\eta(x)$ involving cone invex functions under some suitable assumptions. We also prove the existence of weakly efficient solutions for (MOP) and saddle points for Lagrange function of $(\text{MOP})_\eta(x)$ by using the Karush-Kuhn-Tucker type optimality conditions under generalized convexity functions. As an application, we investigate a multiobjective fractional programming problem by using the modified objective function method.

Keywords Multiobjective optimization problem · Q-(pseudo)invex · Q-convexlike · Weakly efficient solution · Saddlepoint · KKT condition · Weak (strong, converse) duality · Lagrange function

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1 Introduction

The weakly efficient (weak minimum, weak Pareto) solution is an important concept in mathematical models, economics, decision theory, optimal control and game theory (see, for example, [8,12,15,16]). Recently, many authors have studied sufficient and necessary conditions of Karush-Kuhn-Tucker type (for short, KKT) involving weakly efficient solutions for a multiobjective optimization problem (for short, (MOP); see, for example, [10,13] and the references therein). On the other hand, it is well known that duality theory is an important topic in optimization theory. To obtain various duality results various approaches have been proposed (see, for example, [1,6,11,17]).

In most of the papers, some assumptions of convexity were made for the functions in the primal problem. Very recently, some generalized convexity received more attention (see, for example, [9,11,13,14,18,19]). One of them is the invexity introduced by Hanson [13]. In [2], Antczak introduced the modified objective function method for solving a nonlinear (MOP) involving invex functions. He also proved the connection between the weakly efficient solutions for the primal problem and for its modified both objective and constraint functions problem under certain conditions. Li and Li [15] considered a differentiable (MOP) with cone constraints, and showed the existence of weak Pareto solutions for the (MOP) and established the equivalence of weakly efficient solutions for the primal problem and for its η-approximated problem under certain conditions. In [3], Antczak proposed a new approach for solving a scalar nonlinear constrained mathematical programming problem involving invex functions. He pointed out one can obtain optimality conditions and duality results for a nonlinear constrained mathematical programming problem involving invex functions with respect to the same function η by constructing an equivalent minimization problem. Recently, in [5], Antczak used the η-approximation method for obtaining sufficient optimality conditions and duality results for nonlinear mathematical programming problems involving r-invex functions. Antczak [4] used the η-approximation method for obtaining the optimality conditions for a single-ratio objective fractional mathematical programming problem and presented some modified saddle point results for Lagrange function.

Inspired and motivated by above works, the purpose of this paper is to investigate a differentiable (MOP) with generalized cone constraints. We first establish the equivalence between weakly efficient solutions for (MOP) and for the multiobjective optimization problem with the modified objective function and cone constraints $(\text{MOP})_\eta(x)$ and saddle points for the Lagrange function of $(\text{MOP})_\eta(x)$ involving cone invex functions under some suitable assumptions. We also show the existence of weakly efficient solutions for (MOP) and saddle points for Lagrange function of $(\text{MOP})_\eta(x)$ under some KKT conditions. As an application, we investigate a multiobjective fractional programming problem by using the modified objective function method. Here, the modified objective function optimization problem and its Lagrange function for (MOP) are different from those of [6] and [15], respectively.

2 Preliminaries

Let R^n be the n-dimensional Euclidean space and $R^n_+ = \{x = (x_1, \ldots, x_n)^T : x_i \geq 0,\ i = 1, \ldots, n\}$, where the superscript T denotes the transpose. A nonempty subset G of R^n is said to be a cone if $\lambda G \subset G$ for all $\lambda > 0$. G is called a convex cone if G is a cone and $G + G \subset G$.

Throughout this paper, without other specifications, let E be a nonempty open convex subset of R^n, Q be a closed convex cone of R^k with int $Q \neq \emptyset$, S be a closed convex cone of R^m. Let $\eta : E \times E \to R^n$ with $\eta(x, x_0) \neq 0$ for some $x, x_0 \in E$ and $x \neq x_0$. The dual cone of K is denoted by

$$K^* = \{u \in R^n : x^T u \geq 0, \quad \forall x \in K\}.$$

Let $f = (f_1, \dots, f_k)^T : E \to R^k$ and $g = (g_1, \dots, g_m)^T : E \to R^m$. The multiobjective optimization problem (for short, MOP) is defined as follows:

$$\text{(MOP)} \qquad \min \quad f(x)$$
$$\text{Subject to } g(x) \in -S.$$

Denote by $F = \{x \in E : g(x) \in -S\}$ the feasible set of (MOP).

Definition 2.1 A point $x_0 \in F$ is said to be a weakly efficient solution of (MOP) if

$$f(x) - f(x_0) \notin -\text{int}Q, \quad \forall x \in F.$$

We denote by F^w the weakly efficient solutions set of (MOP).

Definition 2.2 [7]

(1) $f : E \to R^k$ is said to be differentiable at $u \in E$ on E if there exists a linear operator L from R^n to R^k such that

$$f(u + h) = f(u) + L(h) + o(\|h\|).$$

(2) The differential L of f at $u \in E$ on E is called the Jacobian operator of f at u on E denoted by $Jf(u)$.
It is well known that f is differentiable at u on E if and only if each f_i, $(i = 1, 2, \dots, k)$ is differentiable at u on E and

$$Jf(u)(h) = (f_1'(u)(h), \dots, f_k'(u)(h))^T, \quad \forall h \in R^n.$$

(3) $f : E \to R^k$ is said to be differentiable on E if it is differentiable at every point of E.

Definition 2.3 (*see, for example*, [2]) Let $\psi : E \to R$ be a differentiable function. ψ is said to be invex with respect to η at $u \in E$ on E if, there exists $\eta : E \times E \to R^n$ such that

$$\psi(x) - \psi(u) - \psi'(u)\eta(x, u) \geq 0, \quad \forall x \in E.$$

Definition 2.4 (*see, for example*, [15]) Let $f : E \to R^k$ be a differentiable function and $P \subset R^k$ a closed convex cone.

(1) f is said to be P-invex with respect to η at $u \in E$ on E if there exists $\eta : E \times E \to R^n$ such that

$$f(x) - f(u) - Jf(u)\eta(x, u) \in P, \quad \forall x \in E.$$

(2) f is said to be P-pseudoinvex with respect to η at $u \in E$ on E if, there exists $\eta : E \times E \to R^n$ such that, for all $x \in E$,

$$Jf(u)\eta(x, u) \notin -\text{int}P \Rightarrow f(x) - f(u) \notin -\text{int}P$$

or, equivalently,

$$f(x) - f(u) \in -\text{int}P \Rightarrow Jf(u)\eta(x, u) \in -\text{int}P.$$

(3) f is said to be P-invex (P-pseudoinvex) with respect to η on E if f is P-invex (P-pseudoinvex) at every point of E with respect to η.
It is easy to see that

$$\text{invexity} \Rightarrow P\text{-invexity} \Rightarrow P\text{-pseudoinvexity}.$$

The following lemma is easy to prove (see, for example, [15]).

Lemma 2.1 *Let K be a convex cone of topological vector space X with $intK \neq \emptyset$. Then, for any $x, y \in X$, the following statements are true:*

(i) $y - x \in K$ *and* $y \in -K$ *imply* $x \in -K$*;*
(ii) $y - x \in K$ *and* $y \in -intK$ *imply* $x \in -intK$*;*
(iii) $y - x \in K$ *and* $x \notin -int\, K$ *imply* $y \notin -intK$*.*

Lemma 2.2 [10] *Let (f, g) be continuously differentiable on $E \times E$. Let $x_0 \in F^w$ and assume that a suitable constraint qualification (CQ) holds. Then there exists $(\lambda, \xi) \in Q^* \times S^*$ with $\lambda \neq 0$ such that*

(i) $\lambda^T Jf(x_0) + \xi^T Jg(x_0) = 0$;
(ii) $\xi^T g(x_0) = 0$.

Remark 2.1 (f, g) is continuously differentiable on $E \times E$ if and only if f and g are continuously differentiable on E, respectively.

Lemma 2.3 [8] *Let $Q \subset R^k$ be a convex cone with int $Q \neq \emptyset$ and Q^* the dual cone of Q. Then we have the following:*

(1) *If $u \in int\, Q$, then $x^T u > 0$ for all $x \in Q^* \setminus \{0\}$;*
(2) *If $x \in int\, Q^*$, then $x^T u > 0$ for all $u \in Q \setminus \{0\}$.*

In [2], Antczak introduced a new method with a modified objective function method for characterization optimality in differentiable nonconvex multiobjective programming problems. An associated vector optimization problem with a modified objective function is constructed in this approach. Now, we give a definition of an vector optimization problem with a modified objective function for the considered in the paper multiobjective optimization problem (MOP).

Let $\bar{x} \in F$. For the original multiobjective programming problem (MOP), we construct the following vector optimization problem with the modified objective function and cone constraints [for short, $(\text{MOP})_\eta(\bar{x})$]:

$$(\text{MOP})_\eta(\bar{x}) \qquad \min \quad Jf(\bar{x})\eta(x, \bar{x})$$
$$\text{Subject to} \quad g(x) \in -S.$$

Denote by F_η^w the weakly efficient solutions set of $(\text{MOP})_\eta(\bar{x})$.

3 Equivalence between (MOP) and $(\text{MOP})_\eta(\bar{x})$

In this section, we investigate the relationship between weakly efficient solutions for (MOP) and for $(\text{MOP})_\eta(\bar{x})$ and discuss saddle points for the Lagrange function of $(\text{MOP})_\eta(\bar{x})$ with generalized convexity functions.

Theorem 3.1 *Let (f, g) be continuously differentiable on $E \times E$ and g be S-invex with respect to η at $\bar{x} \in F$ on E and $\eta(\bar{x}, \bar{x}) = 0$. If $\bar{x} \in F^w$ and a suitable (CQ) holds, then $\bar{x} \in F^w_\eta$.*

Proof Let $\bar{x} \in F^w$ and a suitable (CQ) hold. Then, from Lemma 2.2, there exists $(\lambda, \xi) \in Q^* \times S^*$ with $\lambda \neq 0$ such that

$$\lambda^T Jf(\bar{x}) + \xi^T Jg(\bar{x}) = 0, \quad \xi^T g(\bar{x}) = 0. \tag{3.1}$$

Suppose to the contrary that $\bar{x} \notin F^w_\eta$. Then there exists $\hat{x} \in F$ such that

$$Jf(\bar{x})\eta(\hat{x}, \bar{x}) - Jf(\bar{x})\eta(\bar{x}, \bar{x}) \in -\mathrm{int}Q.$$

Since $\eta(\bar{x}, \bar{x}) = 0$, one has $Jf(\bar{x})\eta(\hat{x}, \bar{x}) \in -\mathrm{int}Q$. It follows from Lemma 2.3 that

$$\lambda^T Jf(\bar{x})\eta(\hat{x}, \bar{x}) < 0. \tag{3.2}$$

Notice that $g(\hat{x}) \in -S$ implies that $\xi^T g(\hat{x}) \leq 0$ and, from (3.1), we get $\xi^T(g(\hat{x}) - g(\bar{x})) \leq 0$. By the S-invexity of g with respect to η at $\bar{x}$ on E, one has

$$g(\hat{x}) - g(\bar{x}) - Jg(\bar{x})\eta(\hat{x}, \bar{x}) \in S$$

and so

$$\xi^T(g(\hat{x}) - g(\bar{x})) - \xi^T Jg(\bar{x})\eta(\hat{x}, \bar{x}) \geq 0.$$

Consequently, we have

$$\xi^T Jg(\bar{x})\eta(\hat{x}, \bar{x}) \leq 0. \tag{3.3}$$

Therefore, it follows from (3.2) and (3.3) that $[\lambda^T Jf(\bar{x}) + \xi^T Jg(\bar{x})]\eta(\hat{x}, \bar{x}) < 0$, which contradicts (3.1). This completes the proof.

Theorem 3.2 *Let $f : E \to R^k$ be Q-pseudoinvex with respect to η at $\bar{x} \in F$ on E and $\eta(\bar{x}, \bar{x}) = 0$. If $\bar{x} \in F^w_\eta$, then $\bar{x} \in F^w$.*

Proof Let $\bar{x} \in F^w_\eta$. If $\bar{x} \notin F^w$, then there exists $\hat{x} \in F$ such that $f(\hat{x}) - f(\bar{x}) \in -\mathrm{int}Q$. Since f is Q-pseudoinvex with respect to η at $\bar{x}$ on E, one has

$$Jf(\bar{x})\eta(\hat{x}, \bar{x}) - Jf(\bar{x})\eta(\bar{x}, \bar{x}) = Jf(\bar{x})\eta(\hat{x}, \bar{x}) \in -\mathrm{int}Q,$$

which is a contradiction with $\bar{x} \in F^w_\eta$. This completes the proof.

Remark 3.1 The assumption that Q-pseudoinvexity of f with respect to η at $\bar{x}$ on E in Theorem 3.2 can be replaced by Q-invexity of f with respect to η at $\bar{x}$ on E. If $Q = R^k_+$ and $S = R^m_+$, then Theorem 3.2 reduces to Theorem 9 in [2]. Moreover, the conditions of Theorem 3.1 are different from that of Theorem 8 in [2].

Let $\bar{x} \in F$. We associate (MOP) with the Lagrange function of $(\mathrm{MOP})_\eta(\bar{x})$:

$$L_\eta(x, \lambda, \xi) := \lambda^T Jf(\bar{x})\eta(x, \bar{x}) + \xi^T g(x), \quad \forall x \in F,\ \lambda \in Q^*,\ \xi \in S^*.$$

Definition 3.1 A point $(\bar{x}, \bar{\lambda}, \bar{\xi}) \in F \times Q^* \setminus \{0\} \times S^*$ is said to be a saddle point of the Lagrange function L_η if, for any $x \in F$, $\xi \in S^*$, $\lambda \in Q^*$,

$$L_\eta(\bar{x}, \lambda, \xi) \leq L_\eta(\bar{x}, \bar{\lambda}, \bar{\xi}) \leq L_\eta(x, \bar{\lambda}, \bar{\xi}).$$

Theorem 3.3 *Let f be Q-pseudoinvex with respect to η at $\bar{x} \in F$ on E and $\eta(\bar{x}, \bar{x}) = 0$. If $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is a saddle point of L_η, then $\bar{x} \in F^w$.*

Proof Let $(\bar{x}, \bar{\lambda}, \bar{\xi})$ be a saddle point of L_η. Then we have

$$L_\eta(\bar{x}, \lambda, \xi) \leq L_\eta(\bar{x}, \bar{\lambda}, \bar{\xi}), \quad \forall \xi \in S^*, \lambda \in Q^*, \tag{3.4}$$

and

$$L_\eta(x, \bar{\lambda}, \bar{\xi}) \geq L_\eta(\bar{x}, \bar{\lambda}, \bar{\xi}), \quad \forall x \in F. \tag{3.5}$$

It follows from (3.4) that $\xi^T g(\bar{x}) \leq \bar{\xi}^T g(\bar{x})$. Letting $\xi = 0$ in the inequality leads to

$$\bar{\xi}^T g(\bar{x}) \geq 0. \tag{3.6}$$

Since $\bar{\xi}^T g(\bar{x}) \leq 0$, one has, from (3.6), $\bar{\xi}^T g(\bar{x}) = 0$.

If $\bar{x} \notin F^w$, then there exists $\hat{x} \in F$ such that

$$f(\hat{x}) - f(\bar{x}) \in -\text{int}Q. \tag{3.7}$$

Since f is Q-pseudoinvex with respect to η at $\bar{x}$ on E, it follows from (3.7) that $Jf(\bar{x})$ $\eta(\hat{x}, \bar{x}) \in -\text{int}Q$ and hence $\bar{\lambda}^T Jf(\bar{x})\eta(\hat{x}, \bar{x}) < 0$. Therefore, we have

$$\begin{aligned} L_\eta(\hat{x}, \bar{\lambda}, \bar{\xi}) &= \bar{\lambda}^T Jf(\bar{x})\eta(\hat{x}, \bar{x}) + \bar{\xi}^T g(\hat{x}) \\ &< \bar{\lambda}^T Jf(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\xi}^T g(\hat{x}) \\ &\leq \bar{\lambda}^T Jf(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\xi}^T g(\bar{x}) \\ &= L_\eta(\bar{x}, \bar{\lambda}, \bar{\xi}), \end{aligned}$$

which contradicts (3.5). This completes the proof.

Remark 3.2 The assumption that Q-pseudoinvexity of f with respect to η at $\bar{x}$ on E in Theorem 3.3 can be replaced by Q-invexity of f with respect to η at $\bar{x}$ on E.

Theorem 3.4 *Let (f, g) be continuously differentiable on $E \times E$. Let g be S-invex with respect to η at $\bar{x} \in F$ on E and $\eta(\bar{x}, \bar{x}) = 0$. If $\bar{x} \in F^w$ and a suitable (CQ) holds, then there exist $\bar{\lambda} \in Q^* \backslash \{0\}$ and $\bar{\xi} \in S^*$ such that $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is a saddle point of L_η.*

Proof Let $\bar{x} \in F^w$. From Lemma 2.2, there exists $(\bar{\lambda}, \bar{\xi}) \in Q^* \times S^*$ with $\bar{\lambda} \neq 0$, such that

$$\bar{\lambda}^T Jf(\bar{x}) + \bar{\xi}^T Jg(\bar{x}) = 0, \quad \bar{\xi}^T g(\bar{x}) = 0. \tag{3.8}$$

Since g is S-invex with respect to η at $\bar{x}$ on E,

$$g(x) - g(\bar{x}) - Jg(\bar{x})\eta(x, \bar{x}) \in S, \quad \forall x \in F,$$

and so

$$\bar{\xi}^T (g(x) - g(\bar{x})) - \bar{\xi}^T Jg(\bar{x})\eta(x, \bar{x}) \geq 0. \tag{3.9}$$

Now, both (3.8) and (3.9) imply that $\bar{\xi}^T g(x) \geq \bar{\xi}^T Jg(\bar{x})\eta(x, \bar{x})$ and $\bar{\xi}^T g(x) \geq -\bar{\lambda}^T Jf$ $(\bar{x})\eta(x, \bar{x})$. Moreover, we have

$$\bar{\lambda}^T Jf(\bar{x})\eta(x, \bar{x}) + \bar{\xi}^T g(x) \geq \bar{\lambda}^T Jf(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\xi}^T g(\bar{x})$$

and, for any $\lambda \in Q^*$ and $\xi \in S^*$,

$$\lambda^T Jf(\bar{x})\eta(\bar{x}, \bar{x}) + \xi^T g(\bar{x}) \leq \bar{\lambda}^T Jf(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\xi}^T g(\bar{x}).$$

Therefore, it follows that

$$L_\eta(\bar{x}, \lambda, \xi) \leq L_\eta(\bar{x}, \bar{\lambda}, \bar{\xi}) \leq L_\eta(x, \bar{\lambda}, \bar{\xi}), \quad \forall x \in F, \lambda \in Q^*, \xi \in S^*.$$

This completes the proof. □

4 Existence of weakly efficient solutions for (MOP) and saddle points for L_η

In this section, we consider the existence of weakly efficient solutions for (MOP) and saddle points for L_η under some KKT conditions.

Theorem 4.1 *Let f and g be Q-pseudoinvex and S-invex with respect to η on E, respectively, and let $\bar{x} \in F$. If there exists $(\lambda, \xi) \in Q^* \setminus \{0\} \times S^*$ such that*

$$\lambda^T Jf(\bar{x}) + \xi^T Jg(\bar{x}) = 0, \quad \xi^T g(\bar{x}) = 0, \tag{4.1}$$

then $\bar{x} \in F^w$.

Proof Let $\bar{x} \in F$ and $(\lambda, \xi) \in Q^* \setminus \{0\} \times S^*$ such that (4.1) holds. Suppose to the contrary that $\bar{x} \notin F^w$. Then there exists $\hat{x} \in F$ such that $f(\hat{x}) - f(\bar{x}) \in -\text{int}Q$ and so, by the Q-pseudoinvexity of f, $Jf(\bar{x})\eta(\hat{x}, \bar{x}) \in -\text{int}Q$. It follows from Lemma 2.3 that

$$\lambda^T Jf(\bar{x})\eta(\hat{x}, \bar{x}) < 0. \tag{4.2}$$

Since $g(\hat{x}) \in -S$, one has $\xi^T g(\hat{x}) \leq 0$. Moreover, it follows that

$$\xi^T (g(\hat{x}) - g(\bar{x})) = \xi^T g(\hat{x}) - \xi^T g(\bar{x}) \leq 0. \tag{4.3}$$

From the S-invexity of g, we get $g(\hat{x}) - g(\bar{x}) - Jg(\bar{x})\eta(\hat{x}, \bar{x}) \in S$ and so

$$\xi^T (g(\hat{x}) - g(\bar{x})) - \xi^T Jg(\bar{x})\eta(\hat{x}, \bar{x}) \geq 0,$$

which, together with (4.3), leads to

$$\xi^T Jg(\bar{x})\eta(\hat{x}, \bar{x}) \leq 0. \tag{4.4}$$

Therefore, it follows from (4.2) and (4.4) that

$$(\lambda^T Jf(\bar{x}) + \xi^T Jg(\bar{x}))\eta(\hat{x}, \bar{x}) < 0,$$

which is a contradiction. This completes the proof. □

Corollary 4.2 *Let f and g be Q-invex and S-invex with respect to η on E, respectively, and let $\bar{x} \in F$. If there exists $(\lambda, \xi) \in Q^* \setminus \{0\} \times S^*$ such that*

$$\lambda^T Jf(\bar{x}) + \xi^T Jg(\bar{x}) = 0, \quad \xi^T g(\bar{x}) = 0,$$

then $\bar{x} \in F^w$.

Theorem 4.2 *Let f be differentiable on E and g be S-invex with respect to η on E, and let $\bar{x} \in F$ with $\eta(\bar{x}, \bar{x}) = 0$. If there exists $(\bar{\lambda}, \bar{\xi}) \in Q^* \setminus \{0\} \times S^*$ such that*

$$\bar{\lambda}^T Jf(\bar{x}) + \bar{\xi}^T Jg(\bar{x}) = 0, \quad \bar{\xi}^T g(\bar{x}) = 0, \tag{4.5}$$

then $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is a saddle point of L_η.

Proof Let $\bar{x} \in F$ and $(\bar{\lambda}, \bar{\xi}) \in Q^* \setminus \{0\} \times S^*$ such that (4.5) holds. Suppose to the contrary that $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is not a saddle point of L_η. Then at least one of the following two statements holds:

(1) There exists $(\hat{\lambda}, \hat{\xi}) \in Q^* \times S^*$ such that

$$L_\eta(\bar{x}, \hat{\lambda}, \hat{\xi}) > L_\eta(\bar{x}, \bar{\lambda}, \bar{\xi}). \tag{4.6}$$

(2) There exists $\hat{x} \in F$ such that

$$L_\eta(\bar{x}, \bar{\lambda}, \bar{\xi}) > L_\eta(\hat{x}, \bar{\lambda}, \bar{\xi}). \tag{4.7}$$

If (1) holds, then from $\eta(\bar{x}, \bar{x}) = 0$ and (4.6), we get

$$\hat{\xi}^T g(\bar{x}) > \bar{\xi}^T g(\bar{x}).$$

Since $g(\bar{x}) \in -S$, from (4.5), one has $0 \geq \hat{\xi}^T g(\bar{x}) > \bar{\xi}^T g(\bar{x}) = 0$, which is a contradiction.

If (2) holds, then from $\eta(\bar{x}, \bar{x}) = 0$ and (4.7), we obtain

$$\bar{\xi}^T g(\bar{x}) > \bar{\lambda}^T Jf(\bar{x})\eta(\hat{x}, \bar{x}) + \bar{\xi}^T g(\hat{x}). \tag{4.8}$$

By S-invexity of g with respect to η, one has $g(\hat{x}) - g(\bar{x}) - Jg(\bar{x})\eta(\hat{x}, \bar{x}) \in S$ and so

$$\bar{\xi}^T (g(\hat{x}) - g(\bar{x})) \geq \bar{\xi}^T Jg(\bar{x})\eta(\hat{x}, \bar{x}). \tag{4.9}$$

Now, both (4.8) and (4.9) imply that $[\bar{\lambda}^T Jf(\bar{x}) + \bar{\xi}^T Jg(\bar{x})]\eta(\hat{x}, \bar{x}) < 0$, which contradicts (4.5). This completes the proof.

Example 4.1 Let $E = R^k = R^m = R^n = R^2$ and $Q = S = \{x = (x_1, x_2)^T : x_1 \geq 0, x_2 \leq 0\}$. Let $f(x) := (x_1, -x_2^2)^T$ and $g(x) := (x_1^2 + 2x_1 - 3, -x_2 - 3)^T$. Consider the following problem (MOP):

$$\text{(MOP)} \qquad \begin{array}{l} \min\ f(x) \\ \text{Subject to} \quad g(x) \in -S. \end{array}$$

One can easily verify that $\bar{x} = (-3, -3)^T$ is a weakly efficient solution of (MOP), f and g are Q-invex and S-invex with respect to the same $\eta(x, y) := (x_1 - y_1, x_2 - y_2)^T$ at $\bar{x}$ on E. Let $\bar{\lambda} = (4, -1)^T$ and $\bar{\xi} = (1, -6)^T$. Then $(\bar{\lambda}, \bar{\xi}) \in Q^* \setminus \{0\} \times S^*$, $\bar{\lambda}^T Jf(\bar{x}) + \bar{\xi}^T Jg(\bar{x}) = 0$ and $\bar{\xi}^T g(\bar{x}) = 0$. The Lagrange function of $(\text{MOP})_\eta(\bar{x})$ is

$$L_\eta(x, \lambda, \xi) := \xi_1 x_1^2 + (2\xi_1 + \lambda_1)x_1 + (6\lambda_2 - \xi_2)x_2 + 3\lambda_1 + 18\lambda_2 - 3\xi_1 - 3\xi_2.$$

Simple computation allows that $(\bar{x}, \bar{\lambda}, \bar{\xi})$ is a saddle point of L_η.

5 An application

In this section, we apply the results obtained above to a multiobjective fractional programming problem (for short, (MFP)). Let $f = (f_1, \ldots, f_k)^T : E \to R^k$, $h = (h_1, \ldots, h_m)^T : E \to R^m$ and $q : E \to R$ with $f_i(x) \leq 0$ and $q(x) > 0$ for any $x \in E$. The (MFP) and $(\text{MFP})_\eta(\bar{x})$ are defined as follows:

$$\text{(MFP)} \qquad \begin{array}{l} \min\ \dfrac{f(x)}{q(x)} := \left(\dfrac{f_1(x)}{q(x)}, \ldots, \dfrac{f_k(x)}{q(x)}\right)^T \\ \text{Subject to} \quad h(x) \in -S, \quad \forall x \in E. \end{array}$$

Denote the feasible set and weakly efficient solutions set of (MFP) by $FP^D = \{x \in E : h(x) \in -S\}$ and FP^w, respectively. Let $\bar{x} \in FP^D$.

$$(\text{MFP})_\eta(\bar{x}) \quad \begin{array}{l} \min\ J\left(\dfrac{f(\bar{x})}{q(\bar{x})}\right)\eta(x, \bar{x}) := \left(J\left(\dfrac{f_1(\bar{x})}{q(\bar{x})}\right)\eta(x, \bar{x}), \ldots, J\left(\dfrac{f_k(\bar{x})}{q(\bar{x})}\right)\eta(x, \bar{x})\right)^T \\ \text{Subject to} \quad h(x) \in -S, \quad \forall x \in E. \end{array}$$

Denote the weakly efficient solutions set of (MFP)$_\eta(\bar{x})$ by FP_η^w. The Lagrange function of (MFP)$_\eta(\bar{x})$ is defined as follows:

$$L_\eta(x,\lambda,\xi)=\lambda^T J\left(\frac{f(\bar{x})}{q(\bar{x})}\right)\eta(x,\bar{x})+\xi^T h(x),\quad \forall x\in FP^D,\ \lambda\in R_+^k\setminus\{0\},\ \xi\in S^*.$$

Lemma 5.1 [14] *Let $\varphi: E\to R$ and $p: E\to R$ with $\varphi(x)\le 0$ and $p(x)>0$ for all $x\in E$. If φ and $-p$ are invex with respect to $\eta: E\times E\to E$ at $u\in E$ on E, then $\frac{\varphi}{p}$ is invex with respect to $\tilde{\eta}$ at u on E, where $\tilde{\eta}(x,u)=\frac{p(u)}{p(x)}\eta(x,u)$.*

Proposition 5.2 *Let f_i $(i=1,2,\ldots,k)$ and $-q$ be invex with respect to $\eta: E\times E\to E$ at $u\in E$ on E. Then $\frac{f}{q}$ is R_+^k-invex with respect to $\tilde{\eta}(x,u)=\frac{q(u)}{q(x)}\eta(x,u)$ at u on E.*

Proof Let f_i $(i=1,2,\ldots,k)$ and $-q$ be invex with respect to η at u on E. It follows from Lemma 5.1 that, for each $i=1,2,\ldots,k$,

$$\frac{f_i(x)}{q(x)}-\frac{f_i(u)}{q(u)}-\left(\frac{f_i(u)}{q(u)}\right)'\tilde{\eta}(x,u)\ge 0$$

and hence

$$\frac{f(x)}{q(x)}-\frac{f(u)}{q(u)}-J\left(\frac{f(u)}{q(u)}\right)\tilde{\eta}(x,u)$$
$$=\left(\frac{f_1(x)}{q(x)}-\frac{f_1(u)}{q(u)}-\left(\frac{f_1(u)}{q(u)}\right)'\tilde{\eta}(x,u),\ldots,\frac{f_k(x)}{q(x)}-\frac{f_k(u)}{q(u)}-\left(\frac{f_k(u)}{q(u)}\right)'\tilde{\eta}(x,u)\right)^T\in R_+^k.$$

Therefore, $\frac{f}{q}$ is a R_+^k-invex with respect to $\tilde{\eta}$ at u on E. This completes the proof.

Theorem 5.3 *Let $\left(\frac{f}{q},h\right)$ be continuously differentiable on $E\times E$ and h be S-invex with respect to η at $\bar{x}$ on E. Assume that some suitable (CQ) hold and $\eta(\bar{x},\bar{x})=0$. If f_i $(i=1,2,\ldots,k)$ and $-q$ are invex with respect to η at $u\in E$ on E, then the following statements are equivalent:*

(i) $\bar{x}\in FP^w$;
(ii) $\bar{x}\in FP_\eta^w$;
(iii) *There exist $\bar{\lambda}\in R_+^k\setminus\{0\}$ and $\bar{\xi}\in S^*$ such that $(\bar{x},\bar{\lambda},\bar{\xi})$ is a saddle point of L_η.*

Proof (i) $\Leftrightarrow$ (ii). It follows from Theorem 3.1 that the necessity is obtained. We only need to prove the sufficiency. Let $\bar{x}\in FP_\eta^w$. Then

$$J\left(\frac{f(\bar{x})}{q(\bar{x})}\right)\eta(x,\bar{x})=J\left(\frac{f(\bar{x})}{q(\bar{x})}\right)\eta(x,\bar{x})-J\left(\frac{f(\bar{x})}{q(\bar{x})}\right)\eta(\bar{x},\bar{x})\notin -\mathrm{int}R_+^k,\quad \forall x\in FP^D.$$

Suppose to the contrary that $\bar{x}\notin FP^w$. Then there exists $\hat{x}\in FP^D$ such that

$$\frac{f(\hat{x})}{q(\hat{x})}-\frac{f(\bar{x})}{q(\bar{x})}\in -\mathrm{int}R_+^k. \tag{5.1}$$

It follows from Proposition 5.2 that $\frac{f}{q}$ is R_+^k-invex with respect to $\tilde{\eta}(x,\bar{x})=\frac{q(\bar{x})}{q(x)}\eta(x,\bar{x})$ at $\bar{x}$ on E. Therefore, we have

$$\frac{f(\hat{x})}{q(\hat{x})}-\frac{f(\bar{x})}{q(\bar{x})}-J\left(\frac{f(\bar{x})}{q(\bar{x})}\right)\left(\frac{q(\bar{x})}{q(\hat{x})}\eta(\hat{x},\bar{x})\right)\in R_+^k. \tag{5.2}$$

From (5.1) and (5.2), it follows that

$$J\left(\frac{f(\bar{x})}{q(\bar{x})}\right)\left(\frac{q(\bar{x})}{q(\hat{x})}\eta(\hat{x},\bar{x})\right) \in -\mathrm{int}R_+^k.$$

Since $q(x) > 0$ and $\frac{q(\bar{x})}{q(\hat{x})} > 0$,

$$J\left(\frac{f(\bar{x})}{q(\bar{x})}\right)\eta(\hat{x},\bar{x}) \in -\mathrm{int}R_+^k,$$

which is a contradiction.

(i) $\Leftrightarrow$ (iii). It follows from Theorem 3.4 that the necessity is obtained. We only need to prove the sufficiency. Let $(\bar{x}, \bar{\lambda}, \bar{\xi})$ be a saddle point of L_η. Then we have

$$L_\eta(\bar{x}, \lambda, \xi) \leq L_\eta(\bar{x}, \bar{\lambda}, \bar{\xi}), \quad \forall \lambda \in R_+^k,\ \xi \in S^*,$$

and

$$L_\eta(x, \bar{\lambda}, \bar{\xi}) \geq L_\eta(\bar{x}, \bar{\lambda}, \bar{\xi}), \quad \forall x \in FP^D. \tag{5.3}$$

Thus we have

$$\xi^T h(\bar{x}) \leq \bar{\xi}^T h(\bar{x}) \leq 0.$$

Letting $\xi = 0$ in the above inequality allows that

$$\bar{\xi}^T h(\bar{x}) = 0. \tag{5.4}$$

If $\bar{x}$ is not a weakly efficient solution of (MFP), then there exists $\hat{x} \in FP^D$ such that

$$\frac{f(\hat{x})}{q(\hat{x})} - \frac{f(\bar{x})}{q(\bar{x})} \in -\mathrm{int}R_+^k.$$

Similarly, we have

$$J\left(\frac{f(\bar{x})}{q(\bar{x})}\right)\eta(\hat{x},\bar{x}) \in -\mathrm{int}R_+^k$$

and so

$$\bar{\lambda}^T J\left(\frac{f(\bar{x})}{q(\bar{x})}\right)\eta(\hat{x},\bar{x}) < 0. \tag{5.5}$$

Since $h(\hat{x}) \in -S$, it follows that

$$\bar{\xi}^T h(\hat{x}) \leq 0. \tag{5.6}$$

Therefore, it follows from (5.4)–(5.6) that

$$\begin{aligned} L_\eta(\hat{x}, \bar{\lambda}, \bar{\xi}) &= \bar{\lambda}^T J\left(\frac{f(\bar{x})}{q(\bar{x})}\right)\eta(\hat{x},\bar{x}) + \bar{\xi}^T h(\hat{x}) \\ &< \bar{\lambda}^T J\left(\frac{f(\bar{x})}{q(\bar{x})}\right)\eta(\bar{x},\bar{x}) + \bar{\xi}^T h(\bar{x}) \\ &= L_\eta(\bar{x}, \bar{\lambda}, \bar{\xi}), \end{aligned}$$

which contradicts (5.3). This completes the proof. □

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